All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the h, k-lemma. Find h, k such that $20h + 49k = 1$. Hence find an integer x such that $x \equiv 4 \pmod{20}$ and $x \equiv 3 \pmod{49}$.

   (b) Prove that there are infinitely many primes of the form $4n + 3$.

   (c) Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. Factorize 3737 into primes in $\mathbb{Z}$ and in $\mathbb{Z}[i]$. Hence find two solutions to $3737 = x^2 + y^2$ $(x, y \in \mathbb{N})$.

2. Let H be a subset of a group G. Give necessary and sufficient conditions for H to be a subgroup of G. In each of the following cases, determine whether or not H is a subgroup of $G = GL_2(\mathbb{R})$, the group of 2×2 invertible real matrices under multiplication, justifying your answer:

   (i) $H = \{A \in G : A^2 = I\}$;

   (ii) $H = \{A \in G : AA^T = I\}$;

   (iii) $H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in G : \alpha, \beta \in \mathbb{R}, \alpha > 0, \beta \geq 0 \right\}$;

   (iv) $H = \{A \in G : AP = PA\}$, where $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

3. (a) State and prove Lagrange's Theorem.

   (b) Prove that if H and K are subgroups of a group G, then so is $H \cap K$.

   (c) Let G be a group of order 15, with H and K subgroups of order 3 and 5 respectively. Prove that $H \cap K = \{e\}$ and deduce that every element of G can be written uniquely in the form hk, $h \in H, k \in K$.

4. (a) Let $A = (a_{ij})$ be an $n \times n$ matrix. Give the definition of $\det(A)$.

(b) Prove that $\det(A^T) = \det(A)$.

(c) Let $x_1, \ldots, x_n$ be real variables. Let $A = (a_{ij})$ be the $n \times n$ matrix defined by $a_{ij} = x_j^{i-1}$. Prove that $\det(A) = \Pi_{j>i}(x_j - x_i)$.

(d) Let $x_1, \ldots, x_n$ be the variables from part (c). Let $B = (b_{ij})$ be the $n \times n$ matrix defined by $b_{ij} = \sum_{k=1}^{n} x_k^{i+j-2}$. Using parts (b) and (c) or otherwise, prove that $\det(B) \geq 0$, with equality only if the x_k $(k = 1, 2, .., n)$ are not all distinct.

5. Let $A = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n (for positive integers n).

(iii) Find a 2×2 matrix X such that $X^2 - 3X - A = 0$.

6. (a) Let A be an $n \times n$ matrix over $\mathbb{R}$. Give the definition of:

(i) an *eigenvalue* λ of A;

(ii) an *eigenvector* $\mathbf{v}$ of A;

(iii) the *characteristic polynomial* $c_A(t)$ of A;

(iv) A is *diagonalizable* (over $\mathbb{R}$).

State the basic criterion for a matrix to be diagonalisable.

(b) Prove that if A has n distinct eigenvalues, then A is diagonalisable.

(c) Let D be a diagonal $n \times n$ matrix with diagonal entries $d_1, .., d_n$, all distinct. Prove that if $XD = DX$ then X is diagonal.

(d) Prove that if A has n distinct eigenvalues and Y and Z are matrices such that $YA = AY$ and $ZA = AZ$ then $YZ = ZY$.

 END OF PAPER